Direction of Arrival Estimation Using Modified Maximum Likelihood Function based on Nyström Method

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***Abstract*—The maximum likelihood (ML) technique offers high performance for the direction-of-arrival (DOA) estimation but is computational expensive. Conventionally, this approach uses the sample covariance matrix (SCM) of the array output. The computation of SCM relies on the array size and available snapshots which consequently leads to a huge computational burden for large array and/or snapshot samples. If calculation of the SCM can be avoided, the reduction of computation complexity is evidently achievable. To circumvent this issue, a modified ML version is proposed. Exploiting the Nyström method allows us to eliminate the SCM computation. The resulting low-rank matrices can be used to construct an accurate signal subspace without calculating the SCM and its eigenvalue decomposition (EVD). Furthermore, the replacement of the SCM by the signal subspace establishes the modified ML function. Regarding to the computation complexity, the complex multiplications between matrices are compared. Several simulation results such as spatial spectrum, root mean squared error (RMSE) and simulation time are included to confirm the tradeoff between the computational time and DOA estimation performance.**

Keywords—direction of arrival, maximum likelihood function, Nyström method

I. Introduction

Direction of arrival (DOA) estimation is one of crucial topics to form spatial spectra in array signal processing. It is extensively used for target positioning in certain fields including wireless communications, seismology, acoustics, radar, sonar etc. In a real-time circumstance, computational time plays an essential role such as the military needs the fastest algorithm to estimate the DOA properly. As a result, one of the challenging issues associated with DOA estimation is to determine the directions of arrival (DOAs) with low complexity and high accuracy as possible.

Spectral-based methods for DOA estimation such as DS [1] (Delay and Sum) and MVDR [2] (Minimum Variance Distortionless Response) are traditional and nonparametric. It offers modest spectral resolutions while requiring large computations due to an inversion of the sample covariance matrix (SCM) of the array output. The computational complexity to compute the SCM and its inversion is $\mathrm{O}(M^2N+M^3)$ where M is the number of antennas element in array and N is the number of snapshots. Subspace-based methods for DOA estimation such as MUSIC [3] (Multiple Signal Classification) and ESPRIT [4] (Estimation Signal Parameter via Rotational Invariance Techniques) are parametric. Its key features provide high spectral resolution, for a large array and/or large samples. However, it is also computationally intensive as it involves the calculation of eigenvalue decomposition (EVD) of SCM to obtain a noise subspace. Regarding the computational complexity of SCM-EVD, the complex multiplications are $\mathrm{O}(M^2N+M^3)$. Model fitting-based methods are also considered to be parametric approaches such as ML [5-6] (Maximum Likelihood) techniques. The ML estimator needs $\mathrm{O}(M^2N+M^3+M^2)$ to perform the SCM-EVD and complex matrix multiplication.

Spectral resolution can be typically improved by increasing the number of array elements or snapshots. Nevertheless, larger numbers come up with expensive computation. In order to achieve low computational complexity and so rapid DOA estimation, various alternatives have been proposed in the literature [7-9]. One technique to circumvent computational bottlenecks when matrices become very large is called Nyström method [10-11] . Based on a low-rank matrix approximation, the use of the Nyström method is considered to be one of the most efficient solutions speeding up the algorithms [12-18]. The algorithms presented in [12-15] exploit the Nyström method to approximate the noise subspace adapted to a low-complexity modified MUSIC version. The noise subspace is constructed without calculating the SCM of the array output and its EVD directly. Accordingly, the simulation time is reduced compared to the MUSIC algorithm under the same DOA efficacy. The Nyström method is applied in [16] to develop a low computational Min-Norm method. By using the Nyström method, a new ESPRIT algorithm requiring much less computational cost is proposed in [17]. Computation burden is even more heavy, especially when the source signals are coherent because the spatial smoothing technology is needed for decorrelation. The algorithms proposed in [18-19] employ the benefit of Nyström method to avoid calculating the SCM of the array output and preprocessing the decorrelation. Compared to the classical spatial smoothing method, the computational complexity is reducible.

Many attempts have been made to devise a low-complexity version for the traditional DOA estimation method. Nonetheless, because of the high computational load of the multivariate nonlinear maximization problem involved, ML estimator has not received considerable attention. In this paper, the ML function with lower complexity and similar performance as the standard ML method is presented. The computational complexity is reduced based on the Nyström method. Moreover, the signal subspace can be formed by the Nyström method. The modification is done by replacing the signal subspace to the SCM in to the standard ML function. Replacement allows us to avoid using the SCM of array output. The proposed version provides the comparable estimation

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performance with the conventional ML method but requires less computational cost and time, particularly in the large array and large samples scenario. It is worth mentioning that we have developed its preliminary version.

II. Array Signal Model

A uniform linear array (ULA) with M omnidirectional antennas can be modelled as depicted in Fig. 1. Assume L narrow-band plane waves with the same center frequency are impinging on the array from distinct directions $\Theta=\{\theta_1,\ldots,\theta_L\}$. Under the restriction that the number of antennas has be greater than the number of the incident signals ($M>L$), estimating unknown angle set Θ is achievable. The noisy array output observed at the snapshot t is expressed as

$$\mathbf{x}(t)=\mathbf{A}(\Theta)\mathbf{s}(t)+\mathbf{n}(t) \qquad t=1,2,\ldots,N \tag{1}$$

where $\mathbf{x}(t)=[x_1(t) \quad \ldots \quad x_M(t)]\in\mathbb{C}^{M\times 1}$ is the array output consisting of incident signals $\mathbf{s}(t)=[s_1(t) \quad \ldots \quad s_L(t)]\in\mathbb{C}^{L\times 1}$ corrupted by a noise vector $\mathbf{n}(t)=[n_1(t) \quad \ldots \quad n_M(t)]\in\mathbb{C}^{M\times 1}$. The signals are steered by the matrix $\mathbf{A}(\Theta)=[\mathbf{a}(\theta_1) \quad \ldots \quad \mathbf{a}(\theta_L)]\in\mathbb{C}^{M\times L}$ whose each column so-called a steering vector due to the l^{th} source toward the angle θ_l given as

$$\mathbf{a}(\theta_l)=[1 \quad e^{j\phi} \quad \ldots \quad e^{j(M-1)\phi}]^T \tag{2}$$

where $\phi=2\pi d\sin(\theta_l)/\lambda$, $\theta_l\in(-\pi/2,\pi/2)$, d is the inter-element spacing; λ is the wavelength and $(\cdot)^T$ is the transpose. The number of samples that the array collects is N snapshots.

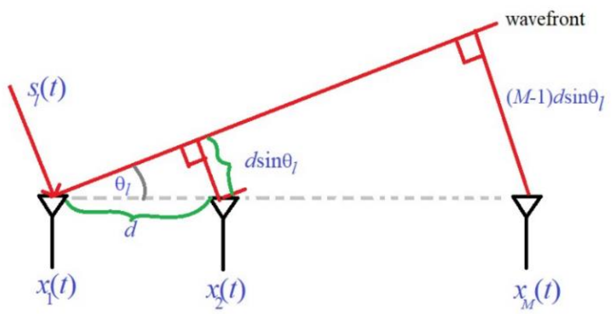

Fig. 1 Uniform linear array (ULA) antenna model

III. DOA Estimation by Maximum Likelihood Function

In general, the concept of the maximum likelihood (ML) estimation is to determine the parameters that make the observed data has the highest joint probability. In the context of DOA estimation, the ML method estimates the angle set by maximize the logarithm of joint probability density function of the array output given in Eq. 1. Due to independent and identically Gaussian distributed noise, the likelihood function of $\mathbf{x}(t)$ can be obtained as [20]

$$L\left(\mathbf{x}(t)\right)=\frac{1}{\pi\det[\sigma_n^2\mathbf{I}_M]}\prod_{t=1}^{N}e^{-\frac{1}{\sigma_n^2}|\mathbf{x}(t)-\mathbf{A}(\Theta)\mathbf{s}(t)|^2} \tag{3}$$

where σ_n^2 is the noise power; $\mathbf{I}_M$ is an $M\times M$ identity and $\det[\cdot]$ stands for the determinant. Instead of maximize the likelihood function, it is simpler to maximize the log-likelihood. Neglecting the constant term in Eq. (3), the log-likelihood function is

$$\ln L=-NM\log\sigma_n^2-\frac{1}{\sigma_n^2}\sum_{t=1}^{N}|\mathbf{x}(t)-\mathbf{A}(\Theta)\mathbf{s}(t)|^2 . \tag{4}$$

Eq. (4) can be maximized if the sum is minimized as

$$\min_{\Theta,\mathbf{s}(t)}\sum_{t=1}^{N}|\mathbf{x}(t)-\mathbf{A}(\Theta)\mathbf{s}(t)|^2 . \tag{5}$$

Fixing Θ, differentiate Eq (5) with respect to $\mathbf{s}(t)$ as

$$\frac{\partial|\mathbf{x}(t)-\mathbf{A}(\Theta)\mathbf{s}(t)|^2}{\partial\mathbf{s}(t)}=2\mathbf{A}(\Theta)|\mathbf{x}(t)-\mathbf{A}(\Theta)\mathbf{s}(t)| . \tag{6}$$

The condition for the occurrence of a maximum point of Eq. (6) is

$$2\mathbf{A}(\Theta)|\mathbf{x}(t)-\mathbf{A}(\Theta)\mathbf{s}(t)|=0 . \tag{7}$$

The source signal can be estimated by solving Eq. (7) as

$$\hat{\mathbf{s}}(t)=\left(\mathbf{A}^H(\Theta)\mathbf{A}(\Theta)\right)^{-1}\mathbf{A}^H(\Theta)\mathbf{x}(t) . \tag{8}$$

Substituting Eq. (8) into Eq. (5) yields an optimization problem as

$$\min_{\Theta}\sum_{t=1}^{N}\left|\mathbf{x}(t)-\mathbf{A}(\Theta)\left(\mathbf{A}^H(\Theta)\mathbf{A}(\Theta)\right)^{-1}\mathbf{A}^H(\Theta)\mathbf{x}(t)\right|^2 \tag{9}$$

Equation (9) can be rewritten in a function of the SCM of the array output as

$$\min_{\Theta}\sum_{t=1}^{N}\left|\left(\mathbf{I}_M-\mathbf{A}(\Theta)\left(\mathbf{A}^H(\Theta)\mathbf{A}(\Theta)\right)^{-1}\mathbf{A}^H(\Theta)\right)\mathbf{x}(t)\right|^2 . \tag{10}$$

$$\min_{\Theta}\; tr\left[\left(\mathbf{I}_M-\mathbf{A}(\Theta)\left(\mathbf{A}^H(\Theta)\mathbf{A}(\Theta)\right)^{-1}\mathbf{A}^H(\Theta)\right)\mathbf{R}\right] \tag{11}$$

where $tr[\cdot]$ is the trace of the bracket matrix and the matrix $\mathbf{R}\in\mathbb{C}^{M\times M}$ is the SCM of the array output calculated by

$$\mathbf{R}=\frac{1}{N}\sum_{t=1}^{N}\mathbf{x}(t)\mathbf{x}^H(t) . \tag{12}$$

Then, DOAs can be estimated by searching an angle set that minimizes Eq. (11). Regarding to fast computation, Eq. (11) essentially relies on the sample covariance matrix which is computationally demanding, especially when the numbers of array elements and snapshots are large. As we can see that the complex multiplication in Eq. (12) depends on the size of array and snapshots.

IV. DOA ESTIMATION BY MODIFIED MAXIMUM LIKELIHOOD FUNCTION BASED ON NYSTRÖM METHOD

When the matrices in Eq. (11) is larger, computational bottlenecks can arise. In practice, fast and efficient calculation is so crucial. Lately, attempts have been made in order to circumvent these bottlenecks. One technique is to use the Nyström method to reduce the computational complexity. In this work, the Nyström method is used to create the signal subspace as the following procedure [13,17].

1. Concatenate the array output vector in to a matrix as $\mathbf{X} = [\mathbf{x}(1) \quad \ldots \quad \mathbf{x}(N)] \in \mathbb{C}^{M\times N}$.
2. Choose a value of $K : \{K \mid L \le K \le \min(M, N)\}$.
3. Split the matrix $\mathbf{X}$ into two submatrices as $\mathbf{X} = \begin{bmatrix} \mathbf{X}_1 \\ \mathbf{X}_2 \end{bmatrix}$ where $\mathbf{X}_1 \in \mathbb{C}^{K\times N}$ and $\mathbf{X}_2 \in \mathbb{C}^{(M-K)\times N}$.
4. Form the matrices:
$$\mathbf{R}_{11} = \frac{1}{N}[\mathbf{X}_1\mathbf{X}_1^H], \ \ \mathbf{R}_{21} = \frac{1}{N}[\mathbf{X}_2\mathbf{X}_1^H].$$
5. Decompose $\mathbf{R}_{11}$ as $\mathbf{R}_{11} = \mathbf{U}_{11}\mathbf{\Lambda}_{11}\mathbf{U}_{11}^H$ where $\mathbf{U}_{11} \in \mathbb{C}^{K\times K}$ is the eigenvector matrix and $\mathbf{\Lambda}_{11} \in \mathbb{C}^{K\times K}$ is the eigenvalues matrix in descend.
6. Compose the matrix $\mathbf{U}_{21} = \mathbf{R}_{21}\mathbf{U}_{11}\mathbf{\Lambda}_{11}^{-1}$.
7. Reduce the size of $\mathbf{X} \in \mathbb{C}^{M\times N}$ by construct $\mathbf{U} \in \mathbb{C}^{M\times K}$ as
$$\mathbf{U} = \begin{bmatrix} \mathbf{U}_{11} \\ \mathbf{U}_{21} \end{bmatrix}.$$
8. Create the matrix $\mathbf{G} = \mathbf{U}\mathbf{\Lambda}_{11}^{1/2}$ and $\mathbf{G}^H\mathbf{G}$.
9. Compute $\mathbf{G}^H\mathbf{G} = \mathbf{U}_G\mathbf{\Lambda}_G\mathbf{U}_G^H$ where $\mathbf{U}_G$ and $\mathbf{\Lambda}_G$ represents the eigenvector and the eigenvalue matrices, respectively.
10. Denote $\mathbf{D} = \mathbf{G}\mathbf{U}_G$ where $\mathbf{D} \in \mathbb{C}^{M\times K}$.
11. Obtain the signal subspace $\mathbf{\Phi}_s \in \mathbb{C}^{M\times L}$ by taking the first L column vectors as $\mathbf{\Phi}_s = \mathbf{D}(:,1:L)$. It is important to mention that $\mathbf{\Phi}_s$ does not depends on the number of snapshots as same as the SCM in Eq. (12).

Once we have the signal subspace in step 11, the modified ML function based on Nyström method is proposed as

$$f(\theta) = tr\left[\left(\mathbf{I}_M - \mathbf{a}(\theta)\mathbf{a}^H(\theta)\right)\mathbf{\Phi}_s\mathbf{\Phi}_s^H\right]. \tag{13}$$

Note that the term $\left(\mathbf{a}^H(\theta)\mathbf{a}(\theta)\right)^{-1}$ in Eq. (11) can be omitted. Finally, the DOAs can be estimated by seeking the angles θ belong to L peaks of Eq. (13).

V. COMPUTATIONAL COMPLEXITY

This section presents the computational complexity analysis by considering the complex multiplication between matrices. The proposed ML method does not require the SCM used in Eq. (12). Instead, it only needs to compute $\mathbf{R}_{11}$ and $\mathbf{R}_{21}$ which require $\mathrm{O}(K^2N)$ and $\mathrm{O}(MNK - K^2N)$, respectively. Meanwhile, the Nyström-based approach just needs $\mathrm{O}(K^2M)$ to construct the signal subspace [17]. The complex matrix multiplication in function (13) takes $\mathrm{O}(M^2L + M^2)$. Consequently, the proposed method requires $\mathrm{O}(MNK + K^2M + M^2L + M^2)$. Meanwhile, the classical ML algorithm needs $\mathrm{O}(M^2N + M^3 + M^2)$ which is much larger due to the cubic term of M and a small value of $K \le \min(M, N)$.

VI. SIMULATION RESULTS

A. Spatial Spectrum

In the first simulation, the performance for DOA estimation is evaluated by comparing the conventional and proposed ML functions. A ULA model with 20 antennas with inter-element spacing $d = \lambda/2$ is set. Two sources with the equal power are impinging from the directions $[-20^\circ, 20^\circ]$. The number of snapshots is $N = 100$. The signal-to-noise ratio (SNR) is defined as the ratio of the power of the source signals to that of the additive noise. The noise is the zero-mean Gaussian process. The spatial spectra of the modified ML function based on the Nyström method compared to the conventional ML method are plotted in Figs. 2-3. The SNR is set to be -10 dB in Fig. 2 whereas the SNR is set to be 0 dB in Fig. 3. Both cases use $K = 8$. It can be observed that the proposed approach forms the peaks of spatial spectrum about $[-20^\circ, 20^\circ]$ properly. Two peaks of the two methods are almost coincide especially in the case of SNR = 0 dB. The results reveal that the proposed method still maintains the performance close to conventional ML estimator.

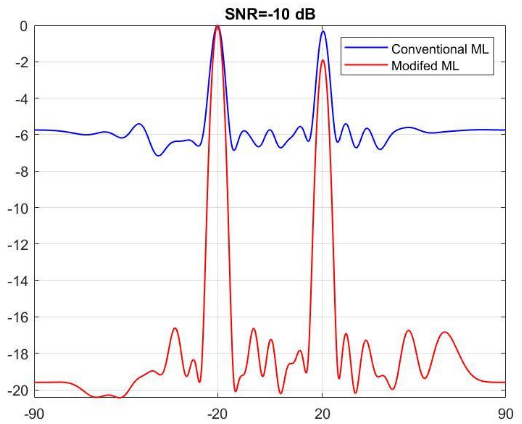

Fig. 2. Comparison of spatial spectra at SNR = -10 dB

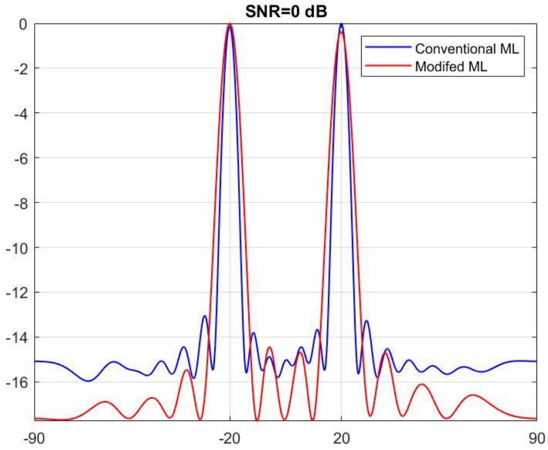

Fig. 3. Comparison of spatial spectra at SNR = 0 dB

B. RMSE versus SNR

In the second simulation, the accuracy of the proposed method is concerned. The accuracy of the DOA estimation is measured by using the root mean squared error (RMSE). To average the squared error, 1000 independent runs are implemented. The RMSE is defined as

$$\text{RMSE}=\sqrt{\frac{1}{1000}\sum_{i=1}^{1000}\left(\hat{\theta}_i-\theta\right)^2} \tag{14}$$

where θ is the actual DOA and $\hat{\theta}_i$ represents the estimated DOA of the i^{th} trial. In the simulation, the ULA with 20 array elements and inter-spacing $d=\lambda/2$ is used to estimate the direction θ. SNR is fixed at 0 dB and user-defined parameter $K=8$ is selected. The RMSE against SNR is depicted in Fig. 4. It can be seen that the proposed algorithm achieves the same performance as the conventional ML method when SNR is more than 10 dB. However, RMSEs for both methods are so close in the case of low SNR.

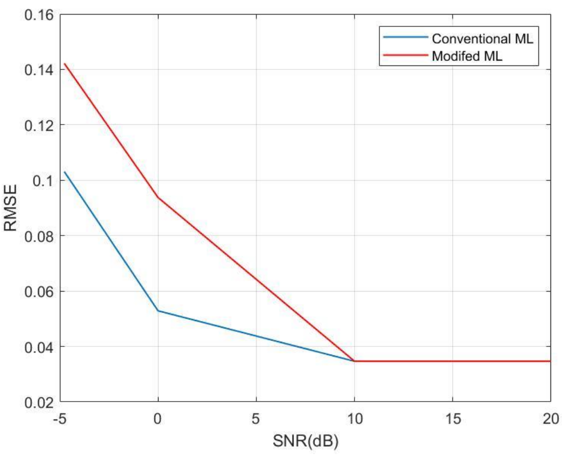

Fig. 4. RMSEs versus SNRs

C. Simulation Time

In the third simulation, the efficiency of DOA estimation is verified by comparing the simulation times against the number of array elements and the number of snapshots as plotted in Figs. 5-6, respectively. The simulation is conducted on a PC with Intel(R) Core(TM) i7-10510U CPU @ 1.80GHz and 8GB RAM. Time recording is done by running the Matlab codes with the tic-toc tool. The SNR is fixed at 0 dB and the other parameters are consistent with the subsection *A*. According to the aim of this paper, the proposed method can reduce the computational time meanwhile maintaining the performance. It can be seen from Figs. 5-6 that the proposed ML version costs a lower simulation time than the standard ML function, especially for large array elements and snapshots. As expected, the computational time increases with the increases of the array elements and snapshots.

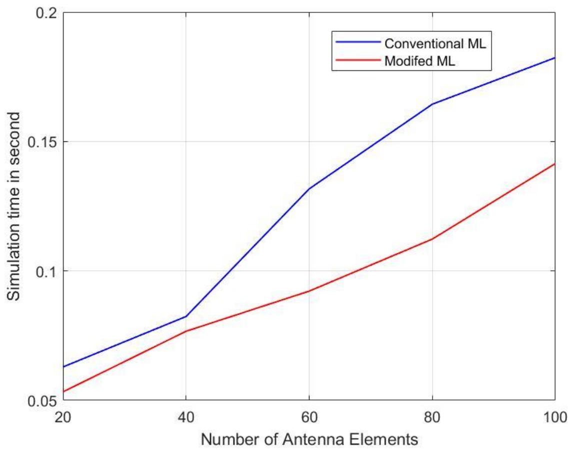

Fig. 5. Simulation time versus the number of antenna elements

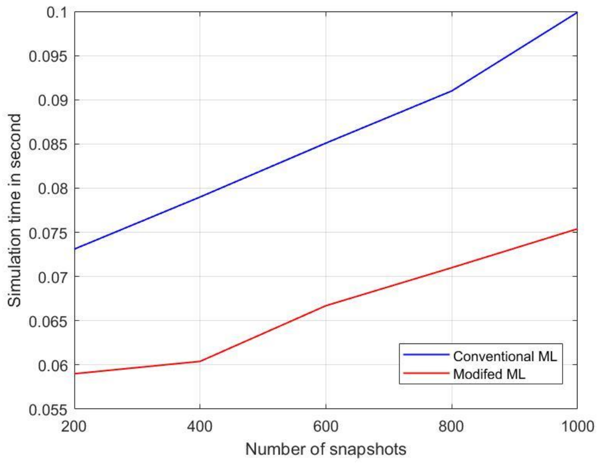

Fig. 6. Simulation time versus the number of snapshots

VII. Conclusions

In this paper, the high computational complexity of the traditional ML method is concerned and a modified version with a lower complexity is proposed. The ML function is modified based on the Nyström method. To decrease the complexity, the Nyström technique is used to construct the signal subspace instead of calculating the SCM of the array output directly. The modification is done by replacing the SCM with the signal subspace. Exploiting the Nyström approximation and modifying the standard ML function can reduce the simulation time while keeping estimation accuracy comparable with the classical ML function. Moreover, the computation complexity analysis is described in term of array size, number of snapshots and number of sources to illustrate that the proposed method can greatly reduce the complexity.

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